Stochastic Processes in Finance - Part I

Jörg Kienitz, Deutsche Postbank AG
email: joerg.kienitz@postbank.de

Abstract
This is the first of a series of articles on stochastic processes in finance. This article covers the key concepts of the theory of stochastic processes used in finance. We work out a stochastic analogue of linear functions and discuss distributional as well as path properties of the corresponding processes. Especially we see that two kinds of movement drive the stochastic component namely diffusion and jumps. Based on this concepts we discuss some financial models heavily used in finance in part II of the series and finally give methods to simulate the processes in part III.

Keywords
Stochastic Processes, Brownian Motion, Poisson Process, Lévy Processes, Fourier Transform, Lévy-Khinchtine Formula

Introduction and Objectives
The Gaussian paradigm in financial engineering is not valid anymore. We will review two kinds of stochastic movement and study the corresponding processes. We identify a class of stochastic processes, namely Lévy processes, which is rich enough to serve as a starting point for financial models and interpret them as a stochastic analogue of a linear function. Using Fourier transform as a useful tool to derive distributional as well as path properties the results are used to analyze financial models and their simulation in the forthcoming two articles.

Generalities on Stochastic Processes
We consider a family \((X_t)_{t \in \mathbb{R}_+}\) of random variables on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). A stochastic process can either be seen as a collection of distributions \((F_t)_{t \in \mathbb{R}_+}\) of the random variables \((X_t)_{t \in \mathbb{R}_+}\) or as a mapping \(t \mapsto X_t(\omega)\) for fixed \(\omega \in \Omega\). In the latter case we call \(X_t(\omega)\) a path.

To further explore stochastic processes, especially regarding the index set as time we have to take care of the information content available in the market and its evolution. Describing the revelation of information with mathematical rigor we need the notion of filtration. A filtration \((\mathcal{F}_t)_{t \in \mathbb{R}_+}\) is a collection of \(\sigma\)-algebras such that \(\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}\) for \(s \leq t\).

The sigma algebra \(\mathcal{F}_t\) can be seen as the information available up to time \(t\). A probability space obeying this property is called filtered probability space. We can then distinguish information currently available in the market form those which is still a possible scenario and therefore random. All this leads to study nonanticipating processes. For a given filtration \((\mathcal{F}_t)_{t \in \mathbb{R}_+}\), i.e. an information structure, such a process is called adapted to the filtration if \(X_t\) is \(\mathcal{F}_t\)-measurable. An important property of a stochastic process is the martingale property. If the process \((X_t)\) possesses this property the best prediction of the future value of \(X_t\) at time \(s < t\), \(E[X_t|X_s]\), is the present value \(X_s\). In finance this property plays an important role since the No-Arbitrage Theory is based on this concept and allows prices of derivative products to be computed as expectations and therefore use methods like Monte Carlo simulation to compute it.

In the sequel we give two examples for stochastic processes. The examples serve as prototypes of different stochastic movement.

Brownian Motion
To treat Brownian motion mathematically, we consider the mapping

\[ W : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R} ; \quad (t, \omega) \mapsto W_t(\omega). \]

and say \((W_t)\) is a Brownian motion if

- (B1) starts at zero, i.e. \(W_0 = 0\) for almost all \(\omega \in \Omega\)
- (B2) has stationary, independent increments
- (B3) the distribution of \(W(t)\) is Gaussian, \(W_t \sim N(0, t)\).
- (B4) the mapping \(t \mapsto W_t(\omega)\) is continuous for almost all \(\omega \in \Omega\)

Let us shortly comment on the property (B2). Take arbitrary time points \(T := \{t_0, t_1, \ldots, t_n\}\) such that \(0 \leq t_0 \leq t_1 \leq \ldots \leq t_n < \infty\). Then (B2) and (B3) state that \((W_{t_i} - W_{t_{i-1}})_{i=1,\ldots,n}\) are independent Gaussian random variables and \(W_{t+h} - W_t\) does not depend on \(h\). The growth of Brownian motion therefore in each time interval \([S, T]\), \(W(T) - W(S) \sim N(0, T - S)\), and therefore only depends on the difference \(T - S\). Adding a volatility \(\sigma\)
between 0 and t.

(P1) starts at zero, N(0) = 0

(P2) has stationary, independent increments

(P3) the distribution of N(T) is Poisson, i.e. N(t) ∼ Poisson(λt).

(P4) the mapping t ↦ N(t) is piecewise constant and increases by jumps of size λ.

The Poisson process does not obey the martingale property but it is easy to make it into a martingale by subtracting the mean. The mean and variance are those of a Poisson distributed random variable and are both equal to λt. This leads to the compensated Poisson process which is given by ̃N(t) = N(t) − λt. The classical Poisson process can be obtained by setting λ = 1. A Compound Poisson process with intensity λ and jump size distribution J is a stochastic process given by

\[ X(t) = \sum_{i=1}^{N(t)} Y_i(t) \]  \hspace{1cm} (2)

where Y_i are independent and Y_i ∼ J and N(t) being a Poisson process independent of Y_i, i ∈ N.

Like we observed for Brownian motion the variance is of order dt. But this time we observe that the process moves by big jumps of small probability in the time. Figure 2 gives the distribution and some typical paths for the Poisson process.

changes the distribution to N(0, σ^2(t - S)). This corresponds to the growth condition of a linear function in real analysis. Linear functions can be characterized by equal growth over time intervals with the same length.

It is very easy to compute the expectation and the variance for this process they are 0 and σ^2 dt^2 respectively. Furthermore, let us stress that the volatility represented by the parameter σ can also be interpreted as a property of the path. It represents the quadratic variation, a measure of roughness of the path. Figure 1 shows the distribution as well as typical paths of Brownian motion.

**Poisson Process**

We face a completely different stochastic movement by considering the Poisson process. In contrast to Brownian motion it has discontinuous sample paths. It takes values in \( N \).

We consider a sequence of independent exponentially distributed random variables \((τ_n), τ_n \sim E(λ), λ > 0\). We take the sum \( T_n := \sum_{i=1}^{n} τ_i \) and consider the following mapping

\[ N : \mathbb{R}^+ \times \Omega \rightarrow N; \quad N(t) \mapsto \sum_{n \geq 1} 1_{\{t \geq T_n\}}. \]  \hspace{1cm} (1)

\( N(t) \) is called Poisson process with intensity \( λ \). This process is also known as counting process. It counts the random times \( T_n \) which occur

Figure 1: Illustration of a Brownian motion; distributions and paths.

Figure 2: Illustration of a Poisson process; distributions and paths.
A Useful Tool

As a matter of fact each distribution is determined by its characteristic function, the Fourier transform, and vice versa. We only need the definition of the Fourier transform. For a given distribution $F_X$, it is

$$F(x) := E_{F_X}[\exp(iux)],$$

(3)

If the distribution has a density $f$ with respect to the Lebesgue measure the Fourier transform can be written as

$$F(x) = \int_{\mathbb{R}} \exp(iux)f(u)du.$$  

(4)

Using that the sum of independent identically distributed random variables is the product of the corresponding characteristic functions together with the independent increment property for a stochastic process and a time interval divided into $n$ pieces of length $dt$ gives

$$\hat{F}_X(x) := \prod_{i=1}^{n} \hat{F}_{X_{\Delta t}}(x).$$

(5)

Thus, to compute the Fourier transform of $X_t$ it suffices to compute it for a small time interval of length $dt$.

We will do this for the above considered processes in the sequel.

Let us have a closer look at the distribution of a Brownian motion to derive its characteristic function. We observe that the variance of $W_{\Delta t}$ is of order $\Delta t$. Let $P$ be a probability measure with characteristic function $\hat{P}$. For $P$ being the distribution of $\frac{X_{\Delta t}}{\sqrt{\Delta t}}$ we have $\hat{X}_{\Delta t}(u) = \hat{P}(u\sqrt{\Delta t})$. Since the expectation of $X_t$ is 0 we have $\int \hat{P}(dx) = 0$. Using that $\hat{P}$ is the Fourier transform of a probability measure and expanding it into a Taylor series we obtain

$$\hat{X}_{\Delta t}(u) \approx 1 + \frac{\Delta t}{2} u^2 + O(u^4).$$

(6)

This corresponds to a small movement over small time intervals with high probability which is illustrated in Figure 3.

There is yet another way for a stochastic movement represented by a Poisson process. This is a big move over small time intervals with low probability. To study this movement we derive the characteristic function of a Poisson process. To this end we consider a process which jumps from its current state on the interval $dt$ to a level $x$ with probability $\lambda$ and stays at its current state with probability $1 - \lambda$. Therefore, the characteristic function can be easily computed and is

$$\hat{X}_{\Delta t}(u) = \lambda dt e^{iu} + (1 - \lambda) e^{0} = \lambda dt e^{iu} + (e^{iu} - 1)\lambda dt \approx \exp((e^{iu} - 1)\lambda dt).$$

(7)

Thus, we recover the characteristic function of the Poisson process by setting $x = 1$ and $\lambda = 1$, hence

$$E[e^{iuN(t)}] = \exp(\lambda t(e^{iu} - 1)).$$

(9)

Figure 4 illustrates the calculation and the stochastic movement of the Poisson process.

It is now easy to construct a process having jump size $x$ and jump probability $\lambda$. This corresponds to multiplication of the Poisson process $N(t)$ by $x$ and exchange $t$ by $\lambda t$.

Distributional properties can be derived as well. For example the moments of the distribution can be computed by differentiation. We will give the formulas for a general class of stochastic processes for the four basic moments expectation, variance, skew and kurtosis later.

Lèvy Processes

If we have a closer look on the properties of Brownian motion and Poisson motion we observe that they obey the following similarities:
(L1) starts at 0
(L2) has stationary, independent increments
(L3) has stationary increments, i.e. the distribution of \( X(t + h) - X(t) \) does not depend on \( h \)
(L4) is stochastically continuous, i.e. for all \( \epsilon \), \( \lim_{h \to 0} P(|X_{t+h} - X_t| \geq \epsilon) = 0 \)

The processes does only differ in terms of the distribution, Gauss and Poisson, and the path properties, continuous and piecewise constant. If we take the properties (L1) to (L4) as the definition of a class of stochastic processes, called Lévy processes. Browninan and Poisson processes are members of that class. In fact we can interpret this class as a stochastic analogue of a linear function. The property (L2) can be seen as the stochastic analogue of the equal growth condition. Different linear functions obey different constant growth rates. The stochastic analogues obey different distributions. The stochastic analogue has a variety of different properties which can all be analyzed by studying the characteristic function. We will see that the above considered processes are basic components of the general class as the next section shows.

**The Lévy Khinchine Theorem**

In fact the characteristic function of a general Lévy process can be derived. This is the Lévy Khinchine Theorem. For a fixed Lévy measure \( \nu \) we denote \( h(x) := x1_{|x| \leq 1} \) and \( b \) \( := b + \int h(x) \nu(dx) \). Then we have for the Fourier transform of a Lévy process \( (X_t)_{t \in \mathbb{R}^+} \):

\[
\hat{X}_t(u) = \exp(\left( ib u h - 1/2 u^2 \sigma^2 \right) + \int (e^{iux} - 1 - iux) \nu(dx) )t)
\]

(10)

What does the formula tell us and how does a Lévy process actually move? First of all we see that a Lévy process is determined by fixing the function \( b \), the measure \( \nu \) and the parameter \( \sigma \). A triple \( (b, \sigma, \nu) \) is called Lévy triplet. To further shed light on the above question we have to study all the components which constitute the characteristic function given in the Lévy Khinchine Theorem. Firstly, we interpret the Lévy measure \( \nu \) as a generalization of the measure \( \mathbb{P} \) used for illustrating the stochastic movements from above. By the usual convention we will consider jumps of height bigger than 1 as big jumps. Let us further examine the movement of a general Lévy process. The drift and diffusion part of (10) are the characteristic functions of a linear function and a Brownian motion respectively. Let us have a closer look on the jump component

\[
u(u) := \mathbb{E} [X^u] = \frac{1}{\nu} \frac{\partial^2 \nu(x)}{\partial x^2}(0).
\]

Thus,

\[
\mathbb{E} [X(t)] = t \cdot \left( b + \int_{|x| \geq 1} x \nu(dx) \right) \quad \mathbb{V} [X(t)] = t \cdot \left( \sigma + \int x^2 \nu(dx) \right)
\]

(14)

We remark that the distributional properties are determined by the shape of the Lévy measure for big \( |x| \) whereas the path properties are determined by the small \( |x| \). Thus, a Lévy processes can be constructed using three components. The stochastic components correspond to two ways of stochastic movement discussed above.

The following article focuses on financial models after going through all the theoretical concepts here. We give real life examples from several areas of finance like derivatives pricings and portfolio optimization.

**Conclusions**

After introducing the basic vocabulary to deal with stochastic processes we stressed the fact that such mathematical objects can be interpreted in two ways: Distribution and Path. We focussed on two examples serving as prototypes for stochastic movement and reviewed the concept of

\[
\mathbb{P} \left( \int_{|x| \geq 1} x \nu(dx) \right)
\]

sated Poisson process can also be considered by setting \( h(x) = x \). For a general function \( h \) we also get a process which is \( x \) times a Poisson process compensation by a linear function. But it is only partially compensated if \( 0 < h(x) < x \).

But there are processes which would not lead to definite integrals in (10). These are processes with too big or too many jumps. The jump size too big means mathematically that

\[
\int_{|x| \geq 1} |x| \nu(dx) = \infty.
\]

(12)

Too many small jumps means mathematically that

\[
\int_{|x| \leq 1} x^2 \nu(dx) = \infty.
\]

(13)

Therefore, for such processes we have to cut-off with \( h \). If we now consider the general setting with a function \( h \) then the corresponding jump process could also be seen as a compensated Poisson process perturbed by a linear function but the expectation will only be compensated for a value between 0 and 1.

With the general formula at hand we can find out about the behaviour of sample paths. It is differentiable if and only if \( \sigma = 0 \) and \( \nu \) is a diffusion. Continuity is a property of a diffusion and can only be achieved if \( \nu = 0 \) otherwise there are jumps leading to discontinuities. If we have a jump part then there occur finitely many jumps in finite intervals if \( \nu([-1, 1]) < \infty \). Otherwise there are infinitely many. We only have a path of finite variation if \( \int_{|x| \leq 1} |x| \nu(dx) < \infty \).

Distributional properties like the moments of the distribution of \( X_t, t \in \mathbb{R}^+ \), can be computed by differentiation and are given by

\[
m_k := \mathbb{E} [X^k] = \frac{1}{\nu} \frac{\partial^k \nu(x)}{\partial x^k}(0).
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Thus,
characteristic function. Using this tool information of distributional as well as path properties can be derived in an easy way. For a wide class of stochastic processes, namely Lévy processes, we gave an intuitive interpretation as the analogue of a linear function from real analysis. Finally, we stated the characteristic function of a general Lévy process which gave further insights to the stochastic movement of a Lévy process.

LITERATURE