Modelling and Pricing of Variance Swaps for Stochastic Volatilities with Delay

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Abstract: Variance swaps for financial markets with underlying asset and stochastic volatilities with delay are modelled and priced in this paper. We found some analytical close forms for expectation and variance of the realized continuously sampled variance for stochastic volatility with delay both in stationary regime and in general case. The key features of the stochastic volatility model with delay are the following: i) continuous-time analogue of discrete-time GARCH model; ii) mean-reversion; iii) contains the same source of randomness as stock price; iv) market is complete; v) incorporates the expectation of log-return. We can valuate variance swap with delay both in risk-neutral world and in physical world, which is also one of the main feature of stochastic volatility model with delay. We also present an upper bound for delay as a measure of risk. As applications, we provide two numerical examples using S&P60 Canada Index (1998–2002) and S&P500 Index (1990–1993) to price variance swaps with delay. Variance swaps for stochastic volatility with delay is very similar to variance swaps for stochastic volatility in Heston model, but simpler to model and to price it.

1 Introduction

A stock’s variance is a square of stock’s volatility (or standard deviation) and the stock’s volatility is the simplest measure of stock’s riskless or uncertainty. Formally, the volatility $\sigma_0$ is the annualized standard deviation of the stock’s returns during the period of interest, where the subscript $\text{R}$ denotes the observed or “realized” volatility, and $\sigma^2_0$ is the “realized” variance.

The easy way to trade variance, square of volatility, is to use variance swaps, sometimes called realized variance forward contracts (see Carr and Madan (1998)).

Variance swaps are forward contracts on future realized stock variance, the square of the future volatility. This instrument provides an easy way for investors to gain exposure to the future level of variance.

It is known that the probability distribution of an equity has a fatter left tail and thinner right tail than the lognormal distribution (see

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Wilmott (1999, 2000), Hull (2000), and the assumption of constant volatility $\sigma$ in financial model (such as the original Black-Scholes model (see Black & Scholes (1973)) is incompatible with derivative prices observed in the market.

The latest issue has been addressed and studied in several ways, such as:

(i) Volatility is assumed to be a deterministic function of the time:
$\sigma \equiv \sigma(t)$ (see Wilmott et al. (1995)); Merton (1973) extended the term structure of volatility to $\sigma := \sigma_t$ (deterministic function of time), with the implied volatility for an option of maturity $T$ given by
$\hat{\sigma}^2_T = \frac{1}{T} \int_0^T \sigma^2_u du$;

(ii) Volatility is assumed to be a function of the time and the current level of the stock price $S(t)$: $\sigma \equiv \sigma(t, S(t))$ (see Demeterfi, K., Derman, E., Kamal, M., and Zou, J. (1999));

(iii) Volatility is described by stochastic differential equation with the same source of randomness as stock’s price (see Javaheri A, Wilmott, P., and Haug, E. G. (2002));

(iv) The time variation of the volatility involves an additional source of randomness, besides $W_1(t)$, represented by $W_2(t)$, and is given by
\[ d\sigma(t) = a(t, \sigma(t))dt + b(t, \sigma(t))dW_2(t), \]
where $W_2(t)$ and $W_1(t)$ (the initial Wiener process that governs the price process) may be correlated (see Buff (2002), Hull and White (1987), Heston (1993), Fouque, J.-P., Papanicolaou, G. and Sircar, K. R. (2000));

(v) The volatility depends on a random parameter $x$ such as $\sigma(t) \equiv \sigma(x(t))$, where $x(t)$ is some random process (see Elliott and Swishchuk (2004), Griego and Swishchuk (2000), Swishchuk (2000), Swishchuk et al. (2000), Swishchuk (1995));

(vi) Another approach is connected with so-called uncertain volatility scenario (see Avellaneda, M., Levy, A. and Paras, A. (1995), Buff (2002));

(vii) The volatility $\sigma(t, S_t)$ depends on $S_t := S(t + \theta)$ for $\theta \in [-\tau, 0]$, namely, stochastic volatility with delay (see Kazmerchuk, Swishchuk and Wu (2002));

Demeterfi, K., Derman, E., Kamal, M., and Zou, J. (1999) explained the properties and the theory of both variance and volatility swaps. They derived an analytical formula for theoretical fair value in the presence of realistic volatiltiy skews, and pointed out that volatility swaps can be replicated by dynamically trading the more straightforward variance swap (case (ii) above).

Javaheri A, Wilmott, P. and Haug, E. G. (2002) discussed the valuation and hedging of a GARCH(1,1) stochastic volatility model. They used a general and exible PDE approach to determine the first two moments of the realized variance in a continuous or discrete context. Then they approximate the expected realized volatility via a convexity adjustment (case (iii) above).


Brockhaus and Long (2000) provided an analytical approximation for the valuation of volatility swaps and analyzed other options with volatility exposure (case (iv) above).

In the paper Swishchuk (2004) we found the values of variance and volatility swaps for financial markets with underlying asset and variance that follow the Heston (1993) model. We also studied covariance and correlation swaps for the financial markets. As an application, we provided a numerical example using S&P60 Canada Index to price swap on the volatility (case (iv) above).

Also, in working paper Elliott and Swishchuk (2004) we found value of variance swap for financial market with Markov stochastic volatility (case (v) above).

In working paper Schoutens, W., Simons, E. and Tistaert, J. (2003), authors shows that several advanced equity option models incorporating stochastic volatility (the Heston stochastic volatility model with jumps and without jumps in the stock price process, the Barndorff-Nielsen-Shephard model and Levy models with stochastic time) can be calibrated very nicely to a realistic option surface.

In this paper, we are going to incorporate the case (vii) above to price variance swap for stochastic volatility with delay.

In Kazmerchuk, Swishchuk and Wu (2002a) we found the Black-Scholes formula for security markets with delayed response and in Kazmerchuk, Swishchuk and Wu (2002b) we proposed and studied the continuous-time GARCH model for stochastic volatility with delay.

We note that the work by Mohammed, Arriojas and Pap (2001) devoted to the derivation of a delayed Black-Scholes formula for the (B, S)-securities market using PDE approach. In their paper, the stock price satisfies the following equation:
\[ dS(t) = \mu S(t - a)S(t)dt + \sigma (S(t - b))S(t)dW(t), \]
where $a$ and $b$ are positive constants and $\sigma$ is a continuous function, and the price of the option at time $t$ has the form $F(t, S(t))$. They found Black-Scholes formula for this model.

In the paper we study stochastic volatility model with delay and to price variance swaps. We find some analytical close forms for expectation and variance of the realized continuously sampled variance for stochastic volatility with delay both in stationary regime and in general case. The key features of the stochastic volatility model with delay are the following: i) continuous-time analogue of discrete-time GARCH model; ii) mean-reversion; iii) contains the same source of randomness as stock price; iv) market is complete; v) incorporates the expectation of log-return. We also present an upper bound for delay as a measure of risk. As applications of our analytical solutions, we provide two numerical examples using S&P500 Canada Index (1998–2002) and S&P500 Index (1990–1993) to price variance swaps with delay. Varinace swaps for stochastic volatility with delay is very
similar to variance swaps for stochastic volatility in Heston model, but simpler to model and to price it.

2 Variance Swaps

As indicated in Introduction, variance swaps are forward contracts on future realized stock variance, the square of the future volatility.

The easy way to trade variance is to use variance swaps, sometimes called realized variance forward contracts (see Carr and Madan (1998)). Although options market participants talk about volatility, it is variance, or volatility squared, that has more fundamental significance (see Demeterfi, K., Derman, E., Kamal, M., and Zou, J. (1999)).

A variance swap is a forward contract on annualized variance, the square of the realized volatility. Its payoff at expiration is equal to

\[ N(\sigma^2(S) - K_{var}), \]

where \( \sigma^2(S) \) is the realized stock variance (quoted in annual terms) over the life of the contract,

\[ \sigma^2(S) := \frac{1}{T} \int_0^T \sigma^2(s) ds, \]

where \( K_{var} \) is the delivery price for variance, and \( N \) is the notional amount of the swap in dollars per annualized volatility point squared. The holder of variance swap at expiration receives \( N \) dollars for every point by which the stock’s realized variance \( \sigma^2(S) \) has exceeded the variance delivery price \( K_{var} \). We note that usually \( N = \alpha \), where \( \alpha \) is a converting parameter such as 1 per volatility-square, and I is a long-short index (1 for long and -1 for short).

Valuing a variance forward contract or swap is no different from valuing any other derivative security. The value of a forward contract \( P \) on future realized variance with strike price \( K_{var} \) is the expected present value of the future payoff in the risk-neutral world:

\[ P^* = E_{P^*} \{ e^{-rT}(\sigma^2(S) - K_{var}) \}, \]

where \( r \) is the risk-free discount rate corresponding to the expiration date \( T \), and \( E_{P^*} \) denotes the expectation under the risk-neutral measure \( P^* \).

Thus, for calculating variance swaps we need to know only \( E_{P^*}(\sigma^2(S)) \), namely, mean value of the underlying variance. The realised continuously sampled variance is defined in the following way:

\[ V := \text{Var}(S) := \frac{1}{T} \int_0^T \sigma^2(t) dt. \]

The realised discrete sampled variance is defined as follows:

\[ \text{Var}_n(S) := \frac{n}{(n-1)t} \sum_{i=1}^n \log \frac{S_i}{S_{i-1}}, \]

where we neglected by \( \frac{1}{2} \sum_{i=1}^n \log \frac{S_i}{S_{i-1}} \) since we assume that the mean of the returns is of the order \( \frac{1}{n} \) and can be neglected. The scaling by \( \frac{1}{T} \) ensures that these quantities annualized (daily) if the maturity \( T \) is expressed in years (days).

\[ \text{Var}_n(S) \] is unbiased variance estimation for \( \sigma(t) \). It can be shown that (see Brockhaus & Long (2000))

\[ V := \text{Var}(S) = \lim_{n \to \infty} \text{Var}_n(S). \]

In this paper, we are interested in valuing of variance swap for security markets with stochastic volatility \( \sigma(t, S_t) \) with delay, where \( S_t := S(t - \tau) \), \( \tau > 0 \), and \( S(t) \) is a stock price at time \( t \in [0, T] \).

In this way, a variance swap for stochastic volatility with delay is a forward contract on annualized variance \( \sigma^2_\tau(t, S_t) \). Its payoff at expiration equals to

\[ N(\sigma^2_\tau(S) - K_{var}), \]

where \( \sigma^2_\tau(S) \) is the realized stock variance (quoted in annual terms) over the life of the contract,

\[ \sigma^2_\tau(S) := \frac{1}{T} \int_0^T \sigma^2(u, S(u - \tau)) du, \quad \tau > 0. \]

3 Modeling of Financial Markets with Stochastic Volatility with Delay

In this Section, we recall some notions and facts from the paper Kazmerchuk, Swishchuk and Wu (2002b).

3.1 Model of Financial Markets

The bond (riskless asset) is represented by the price function \( B(t) \) such that

\[ B(t) = B_0 e^{rt}, \quad t \in [0, T], \]

where \( r > 0 \) is the risk-free rate of return.

The stock (risky asset) in our model is the stochastic process \( (S(t))_{t \in [-\tau, T]} \) which satisfies the following SDE:

\[ dS(t) = \mu S(t) dt + \sigma(t, S(t - \tau)) S(t) dW(t), \quad t > 0, \]

where \( \mu \in \mathbb{R} \) is an appreciation rate, volatility \( \sigma > 0 \) is a continuous and bounded function and \( W(t) \) is a standard Wiener process.

The initial data for (1) is defined by \( S(t) = \varphi(t) \) is deterministic function, \( t \in [-\tau, 0] \).

Throughout the paper we note

\[ S_t := S(t - \tau). \]

The discounted stock price is defined by

\[ Z(t) := \frac{S(t)}{B(t)}. \]

Using Girsanov’s theorem, we obtain the following result concerning the change of probability measure in above market. Under the assumption \( \int_0^\infty \left( \frac{x - \mu x}{\sigma^2 x} \right) dx < \infty, \ a.s. \) the following holds.
1) There is a probability measure $P^*$ equivalent to $P$ such that
\[ \frac{dP^*}{dP} := \exp\left\{ \int_0^T \left( \frac{r - \mu}{\sigma(s,S)} - \frac{1}{2} \int_0^s \left( \frac{r - \mu}{\sigma(s,S)} \right)^2 ds \right) dW(s) \right\} \] (4)
is its Radon-Nikodym density.

2) The discounted stock price $Z(t)$ is a positive local martingale with respect to $P^*$, and it is given by
\[ Z(t) = Z_0 \exp\left\{ -\frac{1}{2} \int_0^T \sigma^2(s,S)ds + \int_0^T \sigma(s,S)dW^*(s) \right\}. \]
where
\[ W^*(t) := \int_0^t \frac{r - \mu}{\sigma(s,S)} ds + W(t) \] (5)
is a standard Wiener process with respect to $P^*$.

Remarks: 1. Another form, the process $Z(t)$ can be written in, is
\[ dZ(t) = Z(t)\sigma(t,S) dW^*(t), \]
and for $\ln Z(t)$ we obtain the following equation
\[ d\ln Z(t) = -\frac{1}{2} \sigma^2(t,S)dt + \sigma(t,S)dW^*(t). \]

2. A sufficient condition for the right-hand side of (4) to be martingale with $t$ in place of $T$ is
\[ E \exp\left\{ \frac{1}{2} \int_0^T \left( \frac{r - \mu}{\sigma(t,S)} \right)^2 dt \right\} < \infty. \]

In this way, the only source of randomness in our model for the market consisting of one stock $S(t)$ and the bond $B(t)$ is a standard Wiener process $W(t)$, $t \in [0,T]$, with $T$ denoting the terminal time. This Wiener process generates the filtration $\mathcal{F}_t := \sigma[W(s) : 0 \leq s \leq t]$.

From an intuitive point of view, the filtration generated by $S$ (or $Z$), rather than by $W$, is more natural one, since $S$ is the observed process. The following lemma holds (see Kallsen, Taqqu (1995)).

Lemma 3.1 The $P^*$-completed filtrations generated by either $W$, $W^*$, $S$ or $Z$ all coincide.

Since the initial process $\phi$ is deterministic, we must not worry about this.

Theorem 3.1 Suppose that $Z(t)$ is a martingale under $P^*$. Then the model is complete. The price at time 0 for a given integrable contingent claim $C$ is given by
\[ \phi_0 = E_{P^*} (e^{-\gamma T} C) . \]
and the price of the claim at any time $0 \leq t \leq T$ is given by
\[ \phi_t = E_{P^*} (e^{-\gamma (T-t)} C | \mathcal{F}_t) . \]

Let us show that $S(t) > 0$ a.s. for all $t \in [0,T]$, when $\phi(0) > 0$ a.s. Define the following process:
\[ N(t) := \mu t + \int_0^t \sigma(s,S) dW(s), \quad t \in [0,T] \]
This is a semimartingale with quadratic variation $(N)(t) = \int_0^t \sigma^2(s,S)ds$. Then, from equation (2) we get:
\[ dS(t) = S(t)dN(t), \quad S(0) = \phi(0). \]
This equation has a solution:

\[ S(t) = \varphi(0) \exp \left\{ N(t) - \frac{1}{2}(N(t)) \right\} \]

\[ = \varphi(0) \exp \left\{ \mu t + \int_0^t \sigma(u, S_u) dW(u) - \frac{1}{2} \int_0^t \sigma^2(u, S_u) du \right\}. \]

From this we see that if \( \varphi(0) > 0 \) a.s., then \( S(t) > 0 \) a.s. for all \( t \in [0, T] \).

### 3.2 Continuous-time GARCH model for Stochastic Volatility with Delay

As we have seen, in the risk-neutral world the stock price \( S(t) \) has the dynamics:

\[ dS(t) = rS(t)dt + \sigma(t, S_t) dW^*(t), \]  

(8)

where \( W^*(t) \) was defined in (5). Let us consider the following equation for the variance \( \sigma^2(t, S_t) \):

\[ \frac{d\sigma^2(t, S_t)}{dt} = \gamma V + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^t \sigma(s, S_s) dW(s) \right]^2 - (\alpha + \gamma)\sigma^2(t, S_t). \]  

(9)

Here, all the parameters \( \alpha, \gamma, \tau, V \) are positive constants and \( 0 < \alpha + \gamma < 1 \).

The Wiener process \( W(t) \) is the same as in (2).

Taking into account (7), the equation (9) is equivalent to the following:

\[ \frac{d\sigma^2(t, S_t)}{dt} = \gamma V + \frac{\alpha}{\tau} \left( \ln \left( \frac{S(t)}{S(t-\tau)} \right) \right) - \frac{1}{2} \sigma^2(u, S_u) du \]  

\[ - (\alpha + \gamma)\sigma^2(t, S_t). \]  

(10)

Our first attempt (see Kazmerchuk, Swishchuk and Wu (2002a)) was to introduce a continuous version of GARCH in the following way:

\[ \frac{d\sigma^2(t)}{dt} = \gamma V + \frac{\alpha}{\tau} \ln^2 \left( \frac{S(t)}{S(t-\tau)} \right) - (\alpha + \gamma)\sigma^2(t), \]  

(11)

where all the parameters were inherited from its discrete-time analogue:

\[ \sigma_n^2 = \gamma V + \frac{\alpha}{\tau} \ln^2(S_{n-1}/S_{n-1-\tau}) + (1 - \alpha - \gamma)\sigma_{n-1}^2, \quad l = \frac{\tau}{\Delta}, \]

which, in the special case \( l = 1 \), is a well-known GARCH(1,1) model for stochastic volatility without conditional mean of log-return (see Bollerslev (1986)).

J.-C. Duan remarked that it is important to incorporate the expectation of log-return \( \ln(S(t)/S(t-\tau)) \) into (11), which is explicitly shown in (10). Therefore, the stochastic delay differential equation (9) is a continuous-time analogue of GARCH(1,1) model with incorporating of conditional mean of log-return.

Using risk-neutral measure argument, we obtain from (9):

\[ \frac{d\sigma^2(t, S_t)}{dt} = \gamma V + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^t \sigma(s, S_s) dW^*(s) \right] - (\alpha + \gamma)\sigma^2(t, S_t) \]  

\[ - (\alpha + \gamma)\sigma^2(t, S_t). \]  

(12)

4 Variance Swaps for Stochastic Volatility with Delay

#### 4.1 Key Features of Stochastic Volatility Model with Delay

We assume that the underlying asset \( S(t) \) follows the process

\[ dS(t) = \mu S(t)dt + \sigma(t, S_t) dW(t) \]  

(14)

and the asset volatility is defined as the solution of the following equation:

\[ \frac{d\sigma^2(t, S_t)}{dt} = \gamma V + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^t \sigma(s, S_s) dW(s) \right] - (\alpha + \gamma)\sigma^2(t, S_t). \]  

(15)

The key features of the stochastic volatility model with delay in (15) are the following:

i) continuous-time analogue of discrete-time GARCH model;
ii) mean-reversion;
iii) does not contain another Wiener process;
iv) market is complete;
v) incorporates the expectation of log-return.

In the risk-neutral world, the underlying asset \( S(t) \) follows the process

\[ dS(t) = rS(t)dt + \sigma(t, S_t) dW^*(t) \]  

and the asset volatility is defined then as follows

\[ \frac{d\sigma^2(t, S_t)}{dt} = \gamma V + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^t \sigma(s, S_s) dW^*(s) \right] - (\alpha + \gamma)\sigma^2(t, S_t) \]  

\[ - (\alpha + \gamma)\sigma^2(t, S_t). \]  

(16)

where \( W^*(t) \) is defined in (5).

Let us take the expectations under risk-neutral measure \( P^* \) on both sides of the equation above. Denoting \( v(t) := E_P[\sigma^2(t, S_t)] \), we obtain the following deterministic delay differential equation:

\[ \frac{dv(t)}{dt} = \gamma V + \alpha(\mu - r)^2 + \frac{\alpha}{\tau} \int_{t-\tau}^t v(s) ds - (\alpha + \gamma)v(t). \]  

(17)
Notice that (17) has a stationary solution
\[ v(t) = X = V + \alpha r(\mu - r)^2 / \gamma. \] (18)

4.2 Valuing of Variance Swaps with Delay in Stationary Regime under Risk-Neutral Measure

In this case of risk-neutral measure \( \mathbb{P}^* \) we have
\[ v(t) = \mathbb{E}_\mathbb{P}^*[\sigma^2(t, S_t)] = V + \alpha r(\mu - r)^2 / \gamma. \] (19)

Hence,
\[ \mathbb{E}_\mathbb{P}^*[\text{Var}(S)] = \frac{1}{T} \int_0^T \mathbb{E}_\mathbb{P}^*[\sigma^2(t, S_t)]dt = V + \alpha r(\mu - r)^2 / \gamma. \] (20)

Therefore, from (19) and (20) it follows that the price \( \mathbb{P}^* \) of variance swap for stochastic volatility with delay in stationary regime under risk-neutral measure \( \mathbb{P}^* \) equals to
\[ \mathbb{P}^* = e^{-rt}[V - K + \alpha r(\mu - r)^2 / \gamma]. \]

It is interesting to note that (20) contains parameter \( \mu \) even after risk-neutral valuation. This is because of the delay \( \tau \): if \( \tau = 0 \), then
\[ \mathbb{E}_\mathbb{P}^*[\text{Var}(S)] = V \]
and
\[ \mathbb{P}^* = e^{-rt}[V - K]. \]

4.3 Valuing of Variance Swaps with Delay in General Case

There is no way to write a solution in explicit form for arbitrarily given initial data. But we can understand an approximate behavior of solutions of (17) by looking at its eigenvalues. Let us substitute \( v(t) = X + Ce^{\rho t} \) into (17), where \( X \) is defined in (18). Then, the characteristic equation for \( \rho \) is:
\[ \rho = \frac{\alpha}{\rho(1 - e^{-\rho\tau})} - (\alpha + \gamma), \] (21)
which is equivalent to (when \( \rho \neq 0 \)):
\[ \rho^2 = \frac{\alpha}{\tau} - \frac{\alpha}{\tau} e^{-\rho\tau} - (\alpha + \gamma)\rho. \]
The only solution to this equation (21) is \( \rho \approx -\gamma \), assuming that \( \gamma \) is sufficiently small.

Then, the behavior of any solution is stable near \( X \), and
\[ v(t) \approx X + Ce^{-\gamma t} \] (22)
for large values of \( t \).

Let find variance swap for stochastic volatility with delay in the case
\[ v(t) \approx X + Ce^{-\gamma t} = V + \alpha r(\mu - r)^2 / \gamma + Ce^{-\gamma t}. \] (23)

Since
\[ v(0) = \sigma(0, S(0 - \tau)) = \sigma(0, \phi(-\tau)) = \sigma_0, \]
we can find the value of \( C \) from (22)
\[ C = v(0) - X = \sigma_0^2 - V - \alpha r(\mu - r)^2 / \gamma. \] (24)

In this way, from (23) and (24) obtain
\[ v(t) = \mathbb{E}_\mathbb{P}^*[\sigma^2(t, S_t)] \approx V + \alpha r(\mu - r)^2 / \gamma + (\sigma_0^2 - V - \alpha r(\mu - r)^2 / \gamma)e^{-\gamma t}. \] (25)

Hence,
\[ \mathbb{E}_\mathbb{P}^*[\text{Var}(S)] = \frac{1}{T} \int_0^T \mathbb{E}_\mathbb{P}^*[\sigma^2(t, S_t)]dt \approx \frac{1}{T} \int_0^T [V + \alpha r(\mu - r)^2 / \gamma + (\sigma_0^2 - V - \alpha r(\mu - r)^2 / \gamma)e^{-\gamma t}]dt \]
\[ = V + \alpha r(\mu - r)^2 / \gamma + (\sigma_0^2 - V - \alpha r(\mu - r)^2 / \gamma) \frac{1 - e^{-\gamma T}}{\gamma}. \] (26)

Therefore, from (25) and (26) it follows that the price \( \mathbb{P}^* \) of variance swap for stochastic volatility with delay in the case of risk-neutral measure \( \mathbb{P}^* \) equals to
\[ \mathbb{P}^* = e^{-rt}\left[V - K + \alpha r(\mu - r)^2 / \gamma + (\sigma_0^2 - V - \alpha r(\mu - r)^2 / \gamma) \frac{1 - e^{-\gamma T}}{\gamma}\right]. \]

5 Delay as A Measure of Risk

As we could see from the previous Section (see (26)), in risk-neutral world we have the following expression for \( \mathbb{E}_\mathbb{P}^*[\text{Var}(S)] :\)
\[ \mathbb{E}_\mathbb{P}^*[\text{Var}(S)] = \frac{1}{T} \int_0^T \mathbb{E}_\mathbb{P}^*[\sigma^2(t, S_t)]dt \]
\[ \approx \frac{1}{T} \int_0^T [V + \alpha r(\mu - r)^2 / \gamma + (\sigma_0^2 - V - \alpha r(\mu - r)^2 / \gamma)e^{-\gamma t}]dt \]
\[ = V + \alpha r(\mu - r)^2 / \gamma + (\sigma_0^2 - V - \alpha r(\mu - r)^2 / \gamma) \frac{1 - e^{-\gamma T}}{\gamma}. \] (27)

This expression contains all the information about our model, since it contains all the initial parameters. We note that \( \sigma_0^2 := \sigma^2(0, \phi(-\tau)) \).

The sign of the second term in (27) depends on the relationship between \( \sigma_0^2 \) and \( V + \alpha r(\mu - r)^2 / \gamma \). If \( \sigma_0^2 > V + \alpha r(\mu - r)^2 / \gamma \), then the second term in (27) is positive and \( \mathbb{E}_\mathbb{P}^*[\text{Var}(S)] \) stays above \( V + \alpha r(\mu - r)^2 / \gamma \). It means that variance is high and, hence, risk is high. If \( \sigma_0^2 < V + \alpha r(\mu - r)^2 / \gamma \), then the second term in (27) is negative and \( \mathbb{E}_\mathbb{P}^*[\text{Var}(S)] \) stays below \( V + \alpha r(\mu - r)^2 / \gamma \). It means that variance is low and, hence, risk is low. In this way, the relationship
\[ \sigma^2(0, \phi(-\tau)) = V + \alpha r(\mu - r)^2 / \gamma \]
defines the measure of risk in the stochastic volatility model with delay.

To reduce the risk we need to take into account the following relationship with respect to the delay \( \tau \) (which follows from (27)):
\[ \tau < \frac{(V - \sigma^2(0, \phi(-\tau))\gamma)}{\alpha(\mu - r)^2}. \]
6 Comparison of Stochastic Volatility in Heston Model and Stochastic Volatility with Delay

In the paper Swishchuk, A. (2004) we studied variance and volatility swaps for financial markets with underlying asset and variance that follow the Heston (1993) model.

The underlying asset $S_t$ in the risk-neutral world and variance follow the following model, Heston (1993) model:

\[
\begin{align*}
\frac{dS_t}{S_t} &= r_t dt + \sigma_t dW_t^1 \\
\frac{d\sigma_t^2}{\sigma_t^2} &= k (\theta^2 - \sigma_t^2) dt + \gamma \sigma_t dW_t^2,
\end{align*}
\]

where $r_t$ is deterministic interest rate, $\sigma_0$ and $\theta$ are short and long volatility, $k > 0$ is a reversion speed, $\gamma > 0$ is a volatility of volatility parameter, $w_t^1$ and $w_t^2$ are independent standard Wiener processes.

The Heston asset process has a variance $\sigma_t^2$ that follows Cox-Ingersoll-Ross (1988) process, described by the second equation above.

It was found that mean value for realized variance in Heston model has the following expression

\[
E[V] := E_p[\text{Var}(S)] = \frac{1}{T} \int_0^T E \sigma_t^2 dt
\]

\[
= \frac{1}{T} \int_0^T (e^{-kt}(\sigma_0^2 - \theta^2) + \theta^2) dt
\]

\[
= \frac{1}{kT} (\sigma_0^2 - \gamma^2 - \theta^2) + \theta^2,
\]

where $\sigma_0^2$ is a short variance (initial value), $\theta^2$ is a long variance (long-term variance), $k$ is a reversion speed.

From (27) we have the following expression for realized variance with delay

\[
E[V] := E_p[\text{Var}(S)] = \frac{1}{T} \int_0^T E \sigma^2(t, S_t) dt
\]

\[
= \frac{1}{T} \int_0^T [V + \alpha \tau (\mu - r)^2/\gamma + (\sigma_0^2 - V - \alpha \tau (\mu - r)^2/\gamma)e^{-\gamma t}] dt
\]

\[
= \frac{1}{\gamma^T} (\sigma_0^2 - V - \alpha \tau (\mu - r)^2/\gamma) + [V + \alpha \tau (\mu - r)^2/\gamma],
\]

where $\sigma_0^2 := \sigma^2(0, \phi(-\tau))$ is a short variance (initial value), $V$ is a long variance (long-term variance) and $0 < \gamma < 1$ is a constant, weight of $V$.

If we compare these two models we can see that they are very similar, especially when $\tau = 0$ : for realized variance in Heston model we have

\[
E[V] = \frac{1 - e^{-\gamma T}}{\gamma T} (\sigma_0^2 - \theta^2) + \theta^2,
\]

and for realized variance for stochastic volatility with delay we have

\[
E[V] \approx \frac{1 - e^{-\gamma T}}{\gamma T} (\sigma_0^2(0, \phi(-\tau)) - V - \alpha \tau (\mu - r)^2/\gamma) + [V + \alpha \tau (\mu - r)^2/\gamma]
\]

\[
= \frac{1 - e^{-\gamma T}}{\gamma T} (\sigma_0^2(0, \phi(-\tau)) - V) + V.
\]

The parameter $\gamma > 0$ in (29), the weight of $V$, plays the role of a reversion speed in stochastic volatility model with delay, as well as parameter $k$ in Heston model (see (28)).

Therefore, we can use our continuous-time GARCH model for stochastic volatility with delay in a row with stochastic volatility in Heston model to price the variance swaps.

7 Numerical Example 1: S&P60 Canada Index

In this section, we apply the analytical solutions from Section 4.3 to price the variance swap of the S&P60 Canada index for five years (January 1998–February 2002) (see Theoret, R., Zabre, L. and Rostan, P. (2002)).

In the end of February 2002, we wanted to price the fixed leg of a variance swap based on the S&P60 Canada index. The statistics on log returns S&P60 Canada Index for 5 year (January 1997–February 2002) is presented in Table 1.

From the histogram of the S&P60 Canada index log returns on a 5-year historical period (1,300 observations from January 1998 to February 2002) it may be seen leptokurtosis in the histogram. If we take a look at the graph of the S&P60 Canada index log returns on a 5-year historical period we may see volatility clustering in the returns series. These facts indicate about the conditional heteroscedasticity. A GARCH(1,1) regression is applied to the series and the results is obtained as in the next Table 2.

This table allows to generate different input variables to the volatility swap model.

We use the following relationship

\[
\theta = \frac{\gamma}{\alpha} \frac{dV}{dt},
\]

\[
k = \frac{1 - \alpha - \beta}{dt},
\]

where $\alpha, \beta$ are parameters of the GARCH(1,1) model.

<table>
<thead>
<tr>
<th>Series:</th>
<th>LOG RETURNS S&amp;P60 CANADA INDEX</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sample:</td>
<td>1 1300</td>
</tr>
<tr>
<td>Observations:</td>
<td>1300</td>
</tr>
<tr>
<td>Mean:</td>
<td>0.000235</td>
</tr>
<tr>
<td>Median:</td>
<td>0.0000593</td>
</tr>
<tr>
<td>Maximum:</td>
<td>0.051983</td>
</tr>
<tr>
<td>Minimum:</td>
<td>-0.101108</td>
</tr>
<tr>
<td>Std. Dev.:</td>
<td>0.013567</td>
</tr>
<tr>
<td>Skewness:</td>
<td>-0.665741</td>
</tr>
<tr>
<td>Kurtosis:</td>
<td>7.787327</td>
</tr>
</tbody>
</table>
to calculate the following discrete GARCH(1,1) parameters:

ARCH(1,1) coefficient $\alpha = 0.060445$;
GARCH(1,1) coefficient $\beta = 0.927264$;
GARCH(1,1) coefficient $\gamma = 0.012391$;
the Pearson kurtosis (fourth moment of the drift-adjusted stock return) $\xi = 7.787327$;
long volatility $\theta = 0.05289724$;
short volatility $\sigma_0 = 0.01$;
parameter $V = 0.000235$,
$\mu = 0.000235$, $\tau = 1$ (day).

Parameter $V$ may be found from the expression $V = \frac{C}{1 - \alpha - \beta}$, where $C = 2.58 \times 10^{-6}$ is defined in Table 2. Thus, $V = 0.00020991$;
$dt = 1/252 = 0.003968254$.

Now, applying the analytical solutions (26) for a variance swap maturity $T$ of 1 year, we find the following value:

$$E_T[Var(S)] = V + \frac{\alpha(r - \mu)^2}{\gamma} + \frac{(\sigma_0^2 - V - \alpha(r - \mu)^2/\gamma) 1 - e^{-\gamma T}}{\gamma T}$$

\[
= 0.0002 + (0.0604 \times (0.0002 - 0.02)^2/0.0124) \\
+ (0.0001 - 0.0002 - 0.0604 \times (0.0002 - 0.02)^2/0.0124) \\
\times \frac{1 - e^{-0.0124}}{0.0124} \\
= 0.000125803041.
\]

Repeating this approach for a series of maturities up to 30 years and series of delays up to 30 days we obtain the following plot (see Appendix, Figure 3) of S&P60 Canada Index Variance Swap. Figures 1 and 2 (see Appendix) depicts the dependence of the variance swap with delay on delay and maturity, respectively.

8 Numerical Example 2: S&P500 Index

In this section, we apply the analytical solutions from Section 4.3 to price a swap on the variance of the S&P500 index for four years (1990–1993) (see Kazmerchuk, Swishchuk & Wu (2002b)).

The statistics on log returns S&P500 Index for 4 year (1990–1993) is presented in Table 3.

Using maximum likehood method we obtain the following parameters required:

$\alpha = 0.3828$;
$\beta = 0.1062$;
$\gamma = 0.511$;
the Pearson kurtosis (fourth moment of the drift-adjusted stock return) $\xi = 3.296144083$;
long volatility $\theta = 0.04038144$;
short volatility $\sigma_0$ equals to $0.00796645$;
parameter $V = 0.04038144$;
$\mu = 0.000263$, $r = 0.02$ and $\tau = 14$ (days).
Now, applying the analytical solutions (26) for a swap maturity $T$ of 1 year, we find the following value:

$$E_r \{ \text{Var}(S) \} = V + \frac{aT(\mu - r)^2}{\gamma} + (\sigma_0^2 - V - aT(\mu - r)^2/\gamma) \frac{1 - e^{-\gamma T}}{T \gamma}$$

$$= 0.004038144 + (0.3828 \times 14 \times (0.000263 - 0.02)^2 / 0.511)$$

$$+ (0.000063 - 0.04038144 - 0.3828 \times 14)$$

$$\times (0.000263 - 0.02)^2 / 0.511$$

$$\times \frac{1 - e^{-0.511}}{0.511}$$

$$= 0.00988086882.$$

Repeating this approach for a series of maturities up to 30 years and series of delays up to 30 days we obtain the following plot (see Appendix, Figure 6) of S&P500 Index Variance Swap. Figures 4 and 5 (see Appendix) depicts the dependence of the variance swap with delay on delay and maturity, respectively.

**Remark.** The results of this paper were presented on MITACS Project Workshop “Modelling Trading and Risk in the Market”, BIRS, Banff, Alberta, Canada, November 12–13, 2004.

### 9 Conclusions

In the paper we studied stochastic volatility model with delay to model variance swaps. We found some analytical close forms for expectation and variance of the realized continuously sampled variance for stochastic volatility with delay both in stationary regime and in general case. The key features of the stochastic volatility model with delay are the following:

i) continuous-time analogue of discrete-time GARCH model;

ii) mean-reversion;

iii) contains the same source of randomness as stock price;

iv) market is complete; v) incorporates the expectation of log-return. We also presented an upper bound for delay as a measure of risk. As an application of our analytical solutions, we provided two numerical examples using S&P60 Canada Index (1998–2002) and S&P500 Index (1990–1993) to price variance swaps.

## 10 Appendix: Figures

Figure 1 depicts the dependence of Variance Swap with Delay on delay for S&P60 Canada Index (1998–2002).

Figure 2 depicts the dependence of Variance Swap with Delay on maturity for S&P60 Canada Index (1998–2002).

Figure 3 depicts the Variance Swap with Delay on maturity and delay for S&P60 Canada Index (1998–2002).

Figure 4 depicts the dependence of Variance Swap with Delay on delay for S&P500 Index (1990–1993).

Figure 5 depicts the dependence of Variance Swap with Delay on maturity for S&P500 Index (1990–1993).

Figure 6 depicts the Variance Swap with Delay on maturity and delay for S&P500 Index (1990–1993).
Figure 2: Dependence of Variance Swap with Delay on Maturity (S&P60 Canada Index).

Figure 3: Variance Swap with Delay for S&P60 Canada Index.

Figure 4: Dependence of Variance Swap with Delay on Delay (S&P500 Index).

Figure 5: Dependence of Variance Swap with Delay on Maturity (S&P500 Index).
REFERENCES


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