

Option Pricing with Jumps

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Abstract

This paper discusses European option pricing under various discontinuous conditions: option and underlying prices as well as volatility and drift coefficients experience breaks. We consider vanilla and double-barrier options under double-exponential jump diffusion model with jump drift and jump volatility. Our approach consists in applying Laplace transform directly to the pricing equation with further computing option prices and risk parameters via numerical inversion of their Laplace transforms. We focus on simple close-form and quasi-close-form solutions.

1 Introduction

One of the hottest topics in financial mathematics is pricing exotic options consistent with the volatility smile. Pricing problem for exotics can be very complicated even in the standard Black-Scholes model. The Laplace transform effectively solves difficult problems such as pricing barriers, lookback, Asians and other exotics. This has been recognized and there have been made efforts to develop analytical methods for pricing exotics consistently with the smile. Lipton (2002) and Kou (2001) obtained closed-form formulas for barrier and lookbacks under exponential jumps. Here we present an alternative approach to the pricing problem.

We start with constant volatility model and show how to price vanilla, barrier and window double-barrier options. Then we present jump-volatility and double-exponential jump diffusion models and illustrate pricing concepts. Next we consider how to price barriers under both models. Our approach leads to “closed-form” pricing formulas for vanillas and barriers. The models can be calibrated to the vanilla market and used for pricing exotics consistently with the smile.

1.1 Problem Formulation

We assume that the underlying assets evolves by the jump-diffusion process of the form

$$dS_t = (r - d - \lambda m)S_t dt + (e^{J_t} - 1)S_t dN_t(\lambda) + \sigma S_t dW_t, \quad S_0 = S \quad (1.1)$$

where r is risk-free (domestic) interest rate, d is dividend (foreign interest) rate, σ is volatility. $N_t(\lambda)$ is a Poisson process with a constant intensity λ , J_t is a random jump size in the logarithm of the asset price with the PDF $\varpi(J)$, which is independent of the Wiener process. We choose $m = \mathbb{E}(e^J - 1)$ to make the discounted asset price process a martingale.

The double barrier option is characterized by pay-off with strike K , maturity T , upper S_u and lower S_d barrier levels and the corresponding rebates ϕ_u and ϕ_d which can be time dependent. We divide last four quantities by strike K and introduce new variables $x = \ln(S/K)$, $x_u = \ln(S_u/K)$, $x_d = \ln(S_d/K)$.

The value of European double barrier call option $U(t, x)$ satisfies the extended Black-Scholes equation

$$-U_t + \nu U_{xx} + \mu U_x - rU + \lambda \mathbb{E}[U(x+J) - U(x)] = 0 \quad (1.2)$$

subject to the terminal condition and boundary conditions

$$U(x, 0) = \max\{e^x - 1, 0\}; \quad U(x_u, t) = \phi_u(t), \quad U(x_d, t) = \phi_d(t) \quad (1.3)$$

where $\nu = \frac{1}{2}\sigma^2$, $\mu = r - \lambda m - \nu$, $t = T - \tau$.

When it is appropriate we will simplify equation (1.2) applying variable transforms

$$U \rightarrow e^{-rt}U, \quad x \rightarrow x + \mu t. \quad (1.4)$$

This takes our equation to

$$-U_t + \nu U_{xx} + \lambda \mathbb{E}[U(x+J) - U(x)] = 0. \quad (1.5)$$

We note that this variable changes are the transitions to the moving coordinate system. They should be applied with care to bounded problems.

We will need some basic properties of Laplace transform, which are given in appendix A. The last two equations (A.3) is a good reason why Laplace transform became popular during the last decade in option pricing under a probabilistic approach (see H. Geman&M. Yor (1996), A. Lipton

(2002) and a survey by M. Craddock, D. Heath and E. Platen (1999) for details). To calculate the Laplace-transformed prices via taking the expectation could be much easier than to integrate the originals. Here we develop an alternative approach. We Apply Laplace transform directly to PIDE (1.2), initial and boundary conditions (1.3) and obtain

$$\begin{aligned} v\bar{U}_{xx} + \mu\bar{U} - (r+p)\bar{U} + \lambda\mathbb{E}[\bar{U}(x+J) - \bar{U}(x)] &= -\phi(x) \\ \phi(x) = \max\{e^x - 1, 0\}; \bar{U}(x_u) = \bar{\phi}_u, \bar{U}(x_d) &= \bar{\phi}_d. \end{aligned} \quad (1.6)$$

2 Pricing in Laplace Transform Domain

We start with the simplest constant volatility model.

2.1 Vanilla and Double-Barrier Options

Let us consider Laplace transform of a general form one-dimensional heat conduction equation

$$\bar{U}_{xx} + \mu\bar{U}_x - c(p)\bar{U} = -\phi(x) \quad (2.1)$$

subject to $\bar{U}(x_u) = \bar{\phi}_u$ and $\bar{U}(x_d) = \bar{\phi}_d$. The rebates ϕ_u and ϕ_d can depend on time.

The general solution of homogeneous equation (2.1) has the form

$$C_1 e^{k_1 x} + C_2 e^{k_2 x} \quad (2.2)$$

where $k_{1,2}$ are roots of a characteristic equation

$$k^2 + \mu k - c = 0. \quad (2.3)$$

The roots are real and have different signs. We choose $k_1 > 0 > k_2$.

We split our problem to the set of particular problems with only one type of boundary conditions. The simplest form of right side of (2.1) is Dirac delta function

$$\phi(x) = \delta(x - x_0). \quad (2.4)$$

To obtain a solution in this special case, let us integrate (2.1) in the vicinity of x_0 .

$$(\bar{U}_x + \mu\bar{U})|_{x_0-h}^{x_0+h} - \int_{x_0-h}^{x_0+h} c\bar{U}(h)dx = -1. \quad (2.5)$$

Tending h to zero we will see that the continuous function with the break of the first kind for its first derivative satisfies (2.5)

$$\begin{cases} \bar{U}|_{x_0-0} - \bar{U}|_{x_0+0} = 0 \\ \bar{U}_x|_{x_0-0} - \bar{U}_x|_{x_0+0} = 1 \end{cases} \quad (2.6)$$

Plugging (2.2) into (2.6), we obtain that the finite at infinity solution of an unbounded problem is

$$\bar{G}_0(x, x_0) = 2e^{s(x-x_0)} \frac{e^{-q|x-x_0|}}{q} \quad (2.7)$$

where $q = (k_1 - k_2)/2$ and $s = (k_1 + k_2)/2$; s does not depend on Laplace transform variable p .

The integral

$$\bar{W}_0(x) = \int_{-\infty}^{\infty} \bar{G}_0(x, x_0)\phi(x_0)dx_0 \quad (2.8)$$

give us a solution of the original equation (2.1).

Now consider the homogeneous equation (2.1) for semi-infinite regions $x > x_d$ and $x < x_u$. Substituting general solution into upper or lower boundary conditions and taking into account the requirement of finiteness we find that

$$\begin{aligned} \bar{G}_1^d &= \frac{1}{p} e^{k_2(x-x_d)} = \frac{e^{s(x-x_d)} e^{-q(x-x_d)}}{p}, \bar{W}_1^d(x) = p\bar{\phi}_d \bar{G}_1^d(x); \\ \bar{G}_1^u &= \frac{1}{p} e^{k_1(x-x_u)} = \frac{e^{s(x-x_u)} e^{q(x-x_u)}}{p}, \bar{W}_1^u(x) = p\bar{\phi}_u \bar{G}_1^u(x). \end{aligned} \quad (2.9)$$

We divide the finite region problem into two simpler ones introducing functions \bar{G}_2^d and \bar{G}_2^u which have form (2.2) and satisfy the homogeneous equation subject to special boundary conditions

$$\begin{cases} \bar{G}_2^d(x_d) = \bar{G}_2^u(x_u) = \frac{1}{p} \\ \bar{G}_2^d(x_u) = \bar{G}_2^u(x_d) = 0 \end{cases} \quad (2.10)$$

Then the solution that satisfies both boundary functions can be presented as

$$\bar{W}_2^{d,u} = p\bar{\phi}_d \bar{G}_2^d + p\bar{\phi}_u \bar{G}_2^u. \quad (2.11)$$

Simple algebra gives expressions for our fundamental solutions

$$\bar{G}_2^{d,u} = \frac{e^{k_1(x-x_{d,u})} - e^{k_2(x-x_{d,u})}}{p(e^{k_1(x_u-d-x_{d,u})} - e^{k_1(x_d-d-x_{d,u})})} = \pm \frac{e^{s(x-x_{d,u})} \sinh(q(x-x_{d,u}))}{p \sinh(q(x_u-x_d))}. \quad (2.12)$$

The set of fundamental solutions of the first boundary problems $G(x)$ and corresponding solutions of the problems with one-type general form boundary conditions $W(x)$ can be easily combined to provide formulas for a number of different exotic options prices.

We start with the vanilla call option, that has payoff $\phi(x) = \max\{e^x - 1, 0\}$ and $\mu = r$, $c = r + p$. Substituting it into (2.8) and integration yield

$$\bar{U}_{\text{vanilla}} = \begin{cases} \frac{2e^{k_1 x}}{q} \left(\frac{1-k_2}{p} + \frac{k_2}{r+p} \right), & x \leq 0 \\ \frac{2e^{k_2 x}}{q} \left(\frac{1-k_1}{p} + \frac{k_1}{r+p} \right) + \frac{e^x}{p} - \frac{1}{p+r}, & x > 0 \end{cases} \quad (2.13)$$

Now we skip the single barrier problems and consider much more complex for conventional methods double-barrier option pricing. Equation (2.11) presents the solution of a boundary problem with zero terminal conditions. Linear combination of this contract with the vanilla option

that satisfies correct terminal condition

$$\bar{U}_{DB}(x) = p(\bar{\phi}_d - \bar{W}_0(x_d))\bar{G}_2^d(x) + p(\bar{\phi}_u - \bar{W}_0(x_u))\bar{G}_2^u(x) + \bar{W}_0(x) \quad (2.14)$$

provides the solution for wide range of double-out barrier options pricing. We just assumed that the payoff is a known function of underlying price at maturity and the rebate values are known functions of time. For standard double-barrier knock-out options $\bar{\phi}_d = \bar{\phi}_u = 0$, $\bar{W}_0(x) = \bar{U}_{vanilla}(x)$. Additional details and prices for more sophisticated step-barrier options were presented in I. Skachkov (2002).

For numerical results we recommend Stehfest (1970) and Abate&Whitt (1995) algorithms of Laplace transform inversion. Superposition principle given by equation (A.2) presents another way to get numbers from (2.14). All fundamental solutions $G(x)$ have close-form presentations in a time domain. Corresponding formulae could be found in P. Wilmott (1999) and J.C. Hull (1999). H.S. Carslow and J.C. Jaeger (1959) published dozens of option pricing formulae many years before Black-Sholes-Merton equation was derived. If all originals of the transforms in the right side of 2.14 are known, Duhamel's theorem gives that

$$U_{DB}(x, t) = \int_0^t [(\phi_d - W_{0d}) * G_{t_2}^d + (\phi_u - W_{0u}) * G_{t_2}^u] d\tau + W_0(x, t) \quad (2.15)$$

where

$$G_t = \frac{\partial G(x, t)}{\partial t}, f_1 * f_2 = f_1(\tau)f_2(t - \tau). \quad (2.16)$$

2.2 Window Double-Barrier Options

To illustrate the power of our formal approach let us consider the problem that looks like everything but inviting for analytical calculations. For window options we have different boundary conditions for different intervals of time. For example, let us consider the knock-out contract that has two barriers at $0 < t \leq t_1$, has only upper barrier at $t_1 < t \leq t_2$, is unbounded at $t_2 < t \leq t_3$, has only lower barrier at $t_3 < t \leq t_4$, and again has two barriers at $t > t_4$. To simplify notation, we assume without loss of generality that the contracts pays no rebates.

We already have derived the solution for the first period (2.15). Denote this solution as $U_1(x, t)$. The next period has a single upper barrier and defined by (2.9). This contract can be considered as up-and-out option with the payoff equal to $U_1(x, t)$ and maturity date $t - t_1$. The difference between values of an unbounded option with that payoff and upper barrier option that pays nothing at maturity but can be exchanged to the former one when a barrier is touched resolves the problem. In Laplace domain we have

$$\begin{aligned} \bar{U}_2(x) &= \bar{W}_0(x) - p\bar{W}_0(x_u)\bar{G}_1^u(x), \\ \bar{W}_0(x) &= \int_{-\infty}^{\infty} \bar{G}_0(x, x_0)U_1(x_0, t_1)dx_0 = \int_{x_d}^{x_u} \bar{G}_0(x, x_0)U_1(x_0, t)dx_0. \end{aligned} \quad (2.17)$$

In time domain

$$U_2(x, t) = W_0(x, t - t_1) - \int_0^{t-t_1} \frac{\partial G_1^u(x, \tau)}{\partial \tau} W_0(x_u, (t - t_1) - \tau) d\tau. \quad (2.18)$$

This scheme would not be changed for the following periods of time. At the open window period we would have an unbounded solution only, then again the combinations of evolutionary problem solution and a convolution of it with the Green's function of an appropriate boundary problem.

$$\begin{aligned} U_n(x, t) &= W_0(U_{n-1}; x, t - t_{n-1}) \\ &- \int_0^{t-t_{n-1}} \frac{\partial G_n^u(x, \tau)}{\partial \tau} W_0(U_{n-1}; x_B, (t - t_{n-1}) - \tau) d\tau. \end{aligned} \quad (2.19)$$

As usually Green's function approach produces an elegant and intuitive solution. With all kernels of integrals known, we even don't need to jump back and forth between time and Laplace transform spaces. It is the simplest, but not necessarily the most effective computational method. Only Green's function of an unbounded problem is defined by elementary function in time domain. Typically, for one-dimensional parabolic equations they are the infinite series of probability integrals. At the same time, all of them are very "nice" functions in Laplace transform domain.

3 Pricing under the Smile

Now we consider some market models which allow to price vanilla and barrier options consistently with the volatility smile.

3.1 Jump Volatility Model

In this section we model volatility as a Poisson process. Let us assume that the market has two states with the constant volatilities σ_1 and σ_2 . The jump from state "1" to state "2" will be modeled by a Poisson process with intensity λ_1 and intensity λ_2 going the other way (P. Wilmott, 1999).

$$\begin{cases} -U_{1t} + v_1 U_{1xx} + (r - v_1)U_{1x} - rU_1 + \lambda_1(U_2 - U_1) = 0 \\ -U_{2t} + v_2 U_{2xx} + (r - v_2)U_{2x} - rU_2 + \lambda_2(U_1 - U_2) = 0 \end{cases} \quad (3.1)$$

Applying variable transform (it gives us the opportunity to split quadric characteristic equation to two quadratic ones)

$$U \rightarrow e^{-rt}U, x \rightarrow x + rt.$$

and then Laplace transform to (3.1) we obtain

$$v_{1,2}(\bar{U}_{1,2xx} - \bar{U}_{1,2x}) + \lambda_{1,2}(\bar{U}_{2,1} - \bar{U}_{1,2}) - p\bar{U}_{1,2} = -\phi(x) \quad (3.2)$$

subject to

$$\begin{aligned} \phi(x) &= \delta(x - x_0) \text{ unit instantaneous source;} \\ \phi(x) &= \max(e^x - 1, 0) \text{ vanilla call option payoff.} \end{aligned}$$

It is easy to find a close form solution for jump volatility in Laplace transform domain. Let us try a solution in a form

$$\bar{U}_1 = Ce^{kx}, \bar{U}_2 = Be^{kx}$$

Substituting in (3.2) we obtain the system of two characteristic equations

$$\begin{cases} C(k^2 - k - \theta_1 - p/\nu_1) + B\theta_1 = 0 \\ C\theta_2 + B(k^2 - k - \theta_2 - p/\nu_2) = 0 \end{cases} \quad (3.3)$$

where $\theta_i = \lambda_i/\nu_i$. The requirement for the determinant of (3.3) to be zero gives us the system of two quadratic equations that has four real roots on real axis in Laplace transform domain—two of them are positive and two are negative. Sorting them in descending order we get

$$k_{1,4} = \frac{1}{2} \left(1 \pm \sqrt{1 + 4L_1} \right), \quad k_{2,3} = \frac{1}{2} \left(1 \pm \sqrt{1 + 4L_2} \right) \quad (3.4)$$

where

$$L_{1,2} = \frac{1}{2} \left((a+b) \pm \sqrt{(a-b)^2 + 4\theta_1\theta_2} \right), \quad a = \theta_1 + \frac{p}{\nu_1}, \quad b = \theta_2 + \frac{p}{\nu_2}. \quad (3.5)$$

We have 8 arbitrary constants to define, but the system of equations is special and easy to solve. From equation (3.3) we get

$$B_{1,4} = K_1 C_{1,4}, \quad B_{2,3} = K_2 C_{2,3}, \quad K_{1,2} = 1 + (p/\nu_1 - L_{1,2})/\theta_1. \quad (3.6)$$

Four constants are to be defined from the continuity equations (2.6). The system can be divided into two systems of two equations. Straight algebra gives that Green's function for the first state is

$$\bar{G}_{JV}(x, x_0) = C_1 e^{-q_1|x-x_0|+s_1(x-x_0)} + C_2 e^{-q_2|x-x_0|+s_2(x-x_0)} \quad (3.7)$$

where

$$\begin{aligned} C_1 = C_4 &= \frac{2(K_2 - 1)}{q_1(K_2 - K_1)}, \quad C_2 = C_3 = \frac{2(K_1 - 1)}{q_2(K_1 - K_2)} \\ q_1 &= (k_1 - k_4)/2, \quad q_2 = (k_2 - k_3)/2, \quad s_1 = s_2 = 1/2 \end{aligned}$$

The integral

$$\bar{W}_{JV}(x) = \int_{-\infty}^{\infty} \bar{G}_{JV}(x, x_0) \phi(x_0) dx_0 \quad (3.8)$$

gives us a solution of the equation (3.2). Thus for vanilla call option

$$\begin{cases} \bar{U}_1(x) = \frac{1}{p} (C_1 e^{k_1 x} + C_2 e^{k_2 x}), \quad x < 0 \\ \bar{U}_1(x) = \frac{1}{p} (C_3 e^{k_3 x} + C_4 e^{k_4 x} + e^{-x} - 1), \quad x \geq 0 \end{cases} \quad (3.9)$$

For the second state of the market coefficients C_i in (3.9) should be replaced by B_i .

The results above can be generalized to the deterministic time dependent volatility case. We assume exponential relaxation of variance to its long term equilibrium value

$$dv(t) = \kappa(v_\infty - v(t))dt, \quad v(0) = v_0 \quad (3.10)$$

where v_0 and v_∞ are spot and equilibrium diffusion coefficients $\nu = \frac{1}{2}\sigma^2$. Integration yields

$$v(t) = v_\infty \left(1 + \frac{v_0 - v_\infty}{v_\infty} e^{-\kappa t} \right). \quad (3.11)$$

Time transform

$$t \mapsto t - \frac{v_0 - v_\infty}{\kappa v_\infty} (e^{-\kappa t} - 1)$$

gives Black-Scholes equation with constant volatility again (P. Wilmott, 1999).

Two states in economy are characterized by different volatilities and drift rates (K. Osband, 2002-2003). If we could provide perfect delta hedging, drift rate would be eliminated from the Black-Scholes equations and we would have a jump volatility model that we have already analyzed. However regime switches are not observable: we don't know exactly in what state of economy we are. Therefore the possible strategy for dynamic delta hedging of asset diffusion and static hedging of volatility jumps is to choose

$$\delta = Q_1 \bar{U}_{1x} + (1 - Q_1) \bar{U}_{2x}. \quad (3.12)$$

where Q_1 is the expected fraction of time that the system stays in the first state

$$Q_1 = \frac{\lambda_2}{\lambda_1 + \lambda_2}.$$

Jump drift and volatility approximation can be described by a system of two one-dimensional ODEs

$$\begin{cases} \bar{L}_{BS} \bar{U}_1 + (m_1 - r)(1 - Q_1)(\bar{U}_{1x} - \bar{U}_{2x}) + \lambda_1(\bar{U}_2 - \bar{U}_1) = -\varphi(x) \\ \bar{L}_{BS} \bar{U}_2 + (m_2 - r)Q_1(\bar{U}_{2x} - \bar{U}_{1x}) + \lambda_2(\bar{U}_1 - \bar{U}_2) = -\varphi(x) \end{cases} \quad (3.13)$$

where m_1 and m_2 are drift rates in two states,

$$\bar{L}_{BS} \bar{U}_i = \nu_i \bar{U}_{ixx} + (r - \nu_i) \bar{U}_{ix} - (r + p) \bar{U}_i.$$

The actual price of option depends on subjective probability ("believing") P_1 that the market is in the first state.

$$\bar{U} = P_1 \bar{U}_1 + (1 - P_1) \bar{U}_2 \quad (3.14)$$

The dynamics of “beleiving” under regime switching is discussed by K. Osband in Wilmott Magazine. We will consider it as one of the parameters to calibrate jump volatility model.

3.2 Double-Exponential Jump model

We consider double-exponential jumps with PDF given by

$$\varpi(J) = q \frac{1}{\eta^+} e^{-\frac{1}{\eta^+} J} \mathbf{1}_{\{J \geq 0\}} + (1 - q) \frac{1}{\eta^-} e^{\frac{1}{\eta^-} J} \mathbf{1}_{\{J < 0\}} \quad (3.15)$$

where $1 > \eta^+ > 0$, $\eta^- > 0$ are mean sizes of positive and negative jumps, respectively; $q \geq 0$. q and $1 - q$ represent the probabilities of positive and negative jumps, respectively.

The model is also referred to as Kou jump-diffusion model (Kou, 2001). We will need the following properties of this distribution:

$$\mathbb{E}[e^{kJ}] = \int_{-\infty}^{\infty} e^{kJ} \varpi(J) dJ = \frac{q}{1 - k\eta^+} + \frac{1 - q}{1 + k\eta^-}. \quad (3.16)$$

It follows that

$$m = \mathbb{E}[e^J - 1] = \frac{q}{1 + \eta^-} + \frac{1 - q}{1 - \eta^+} - 1.$$

A closed-form solution in Laplace Transform domain for double-exponential jumps model was obtained by Sepp (2003). A problem can be described by the PIDE (1.2). For the expectation of option price under jump process we have

$$\begin{aligned} \mathbb{E}(U(x + J)) &= q \int_{-\infty}^0 U(x + J) \varpi^-(J) dJ + (1 - q) \int_0^{\infty} U(x + J) \varpi^+(J) dJ \\ &= q \int_{-\infty}^x U * \varpi^- dy + (1 - q) \int_{-x}^{\infty} U * \varpi^+ dy \end{aligned} \quad (3.17)$$

where

$$\int_a^b U * \varpi^\pm dy = \int_a^b U(y) \varpi^\pm(y - x) dy.$$

For semi-infinite intervals ($0 < x$) and ($x \geq 0$) we have correspondingly

$$\begin{aligned} L_x \bar{U}^- + \lambda q \int_{-\infty}^x \bar{U}^- * \varpi^- dy \\ + \lambda(1 - q) \left(\int_x^0 \bar{U}^- * \varpi^+ dy + \int_0^{\infty} \bar{U}^+ * \varpi^+ dy \right) = 0, \end{aligned} \quad (3.18)$$

$$\begin{aligned} L_x \bar{U}^+ + \lambda q \left(\int_{-\infty}^+ \bar{U}^- * \varpi^- dy + \int_0^x \bar{U}^+ * \varpi^- dy \right) \\ + \lambda(1 - q) \int_x^{\infty} \bar{U}^+ * \varpi^+ dy = -\varphi(x). \end{aligned} \quad (3.19)$$

where L_x is a differential operator. If drift and volatility are constants it is simply

$$L_x = v \frac{\partial^2}{\partial x^2} + \mu \frac{\partial}{\partial x} - (\lambda + r + p). \quad (3.20)$$

To define a solution of equations (3.18) and (3.19), we try the form

$$\begin{cases} \bar{U}^- = \sum_i C_i^- e^{k_i x} \\ \bar{U}^+ = \sum_i C_i^+ e^{m_i x} + \frac{e^x}{p} - \frac{1}{r+p}. \end{cases} \quad (3.21)$$

Substituting in the equation (1.2) we obtain

$$\begin{aligned} &\sum_i C_i^- \Phi_{k_i} e^{x k_i} \\ &+ \lambda(1 - q) \left(\sum_i \frac{C_i^-}{k_i \eta^+ - 1} - \sum_i \frac{C_i^+}{m_i \eta^- - 1} - \frac{1}{d(\eta^+ - 1)} - \frac{1}{p + r} \right) e^{-\frac{1}{\eta^+} x} = 0, \\ &\sum_i C_i^+ \Phi_{m_i} e^{x m_i} \\ &+ \lambda q \left(\sum_i \frac{C_i^-}{m_i \eta^- + 1} - \sum_i \frac{C_i^+}{m_i \eta^- + 1} - \frac{1}{d(\eta^- - 1)} + \frac{1}{p + r} \right) e^{\frac{1}{\eta^-} x} = 0, \\ &\Phi_k = e^{-kx} L_x e^{kx} + \lambda \left(\frac{q}{1 + \eta^- k} + \frac{1 - q}{1 - \eta^+ k} \right). \end{aligned} \quad (3.22)$$

To satisfy equation (3.21) we should require that all coefficients on different exponents be equal to zero. We have the following characteristic equation

$$vk^2 + \mu k - (\lambda + r + p) + \lambda \left(\frac{q}{1 + \eta^- k} + \frac{1 - q}{1 - \eta^+ k} \right) = 0, \quad (3.23)$$

Equation (3.23) has four different real roots, two positive and two negative. Therefore, we have two equations to define four arbitrary constants.

First two equations and the requirement of the smoothness of the solution and its first derivative at point $x = 0$ yield a system to determine constants in formula (3.21).

As a result, we obtain that the value of a vanilla call under the double-exponential jump-diffusion is given by formula

$$\bar{U} = \begin{cases} C_0 e^{k_0 x} + C_1 e^{k_1 x}, & x < 0 \\ C_2 e^{k_2 x} + C_3 e^{k_3 x} + \frac{e^x}{p} - \frac{1}{r+p}, & x \geq 0 \end{cases} \quad (3.24)$$

where k_0, k_1, k_2 and k_3 are four real roots of characteristic equation (3.23) (for computation purposes it can be presented as a polynomial) such that

$$-\infty < k_3 < -\frac{1}{\eta^-} < k_2 < 0 < k_1 < \frac{1}{\eta^+} < k_0 < \infty \quad (3.25)$$

and constants C_0, C_1, C_2, C_3 are solution of the system

$$\begin{pmatrix} 1 & 1 & -1 & -1 \\ k_0 & k_1 & -k_2 & -k_3 \\ \frac{1}{k_0\eta^-+1} & \frac{1}{k_1\eta^-+1} & -\frac{1}{k_2\eta^-+1} & -\frac{1}{k_3\eta^-+1} \\ \frac{1}{k_0\eta^+-1} & \frac{1}{k_1\eta^+-1} & -\frac{1}{k_2\eta^+-1} & -\frac{1}{k_3\eta^+-1} \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{p} - \frac{1}{r+p} \\ \frac{1}{p} \\ \frac{1}{p(\eta^+-1)} - \frac{1}{r+p} \\ \frac{1}{p(\eta^+-1)} + \frac{1}{r+p} \end{pmatrix}. \quad (3.26)$$

4 Double-Barrier Options under jump volatility and jump diffusion models

Option pricing in areas with fixed boundary conditions is a problem of the same complexity as the pricing in infinite regions. In the section 2 we derived the formula for standard double-barrier options. The same approach can be applied to more sophisticated models than we have developed above:

1. Derive a solution of an unbounded problem in laboratory (still) coordinate system that satisfies initial conditions.
2. Add general solution of homogeneous equation and substitute the sum into boundary conditions to define arbitrary constants.

For the jump volatility model the procedure is obvious. For the jump diffusion model that is described by OIDE we would have four additional constants and two boundary conditions. Two other equations can be obtained from the integral part of equation. We would have two additional intervals in integrals (3.17) where our function is constant and equal to rebate values.

$$\mathbb{E}[\bar{U}(x+J)] = \int_{-\infty}^{x_d} \bar{\varphi}_d * \omega dy + \int_{x_d}^{x_u} \bar{U} * \omega dy + \int_{x_u}^{\infty} \bar{\varphi}_u * \omega dy \quad (4.1)$$

where $x_u > 0$ and $x_d < 0$.

It follows that

$$\mathbb{E}[\bar{U}(x+J)] = \bar{\varphi}_d e^{\frac{1}{\eta^-}(x_d-x)} + \int_{x_d}^{x_u} \bar{U} * \omega dy + \bar{\varphi}_u e^{-\frac{1}{\eta^+}(x_u-x)}. \quad (4.2)$$

Presenting solution as

$$\bar{U} = \bar{U}_\infty + \bar{U}_B \quad (4.3)$$

where \bar{U}_∞ is the solution of unbounded problem and \bar{U}_B is a general solution

$$\bar{U}_B = \sum_{i=1}^4 B_i e^{k_i x} \quad (4.4)$$

and substituting (4.2)-(4.4) into equations (3.18) and (3.19) we get that

$$\sum_{i=1}^4 B_i \frac{e^{k_i x_d}}{k_i \eta^- + 1} = - \sum_{i=1}^2 C_i \frac{e^{k_i x_d}}{k_i \eta^- + 1} + \bar{\varphi}_d \quad (4.5)$$

and

$$\sum_{i=1}^4 B_i \frac{e^{k_i x_u}}{k_i \eta^+ - 1} = - \sum_{i=3}^4 C_i \frac{e^{k_i x_u}}{k_i \eta^+ - 1} - \frac{e^{x_u}}{p(\eta^+ - 1)} - \frac{1}{p+r} - \bar{\varphi}_u. \quad (4.6)$$

Boundary conditions give another pair of equations. Summing up, we obtain that the value of a double-barrier option under the double-exponential jump-diffusion is given by formula

$$\bar{U}^{DB} = \begin{cases} (C_0 + C_4)e^{k_0 x} + (C_1 + C_5)e^{k_1 x} + C_6 e^{k_2 x} + C_7 e^{k_3 x}, & x < 0 \\ (C_2 + C_6)e^{k_2 x} + (C_3 + C_7)e^{k_3 x} + C_4 e^{k_0 x} + C_5 e^{k_1 x} + \left[\frac{e^x}{p} - \frac{1}{r+p} \right], & x \geq 0 \end{cases} \quad (4.7)$$

where constants $C_j, j = 0, \dots, 3$, are solution of system (3.26) and constants $C_j, j = 4, \dots, 7$, are solution of system

$$\begin{pmatrix} \frac{e^{k_0 x_d}}{k_0 \eta^- + 1} & \frac{e^{k_1 x_d}}{k_1 \eta^- + 1} & \frac{e^{k_2 x_d}}{k_2 \eta^- + 1} & \frac{e^{k_3 x_d}}{k_3 \eta^- + 1} \\ e^{k_0 x_u} & e^{k_1 x_u} & e^{k_2 x_u} & e^{k_3 x_u} \\ \frac{e^{k_0 x_u}}{k_0 \eta^+ - 1} & \frac{e^{k_1 x_u}}{k_1 \eta^+ - 1} & \frac{e^{k_2 x_u}}{k_2 \eta^+ - 1} & \frac{e^{k_3 x_u}}{k_3 \eta^+ - 1} \end{pmatrix} \begin{pmatrix} C_4 \\ C_5 \\ C_6 \\ C_7 \end{pmatrix} = \begin{pmatrix} \bar{\varphi}_d - \frac{e^{k_0 x_d} C_0 - e^{k_1 x_d} C_1}{k_0 \eta^- + 1} \\ \bar{\varphi}_d - e^{k_0 x_d} C_0 - e^{k_1 x_d} C_1 \\ -\frac{e^{x_u}}{p} + \frac{1}{r+p} + \bar{\varphi}_u - e^{k_2 x_u} C_2 - e^{k_3 x_u} C_3 \\ -\frac{e^{x_u}}{p(\eta^+ - 1)} - \frac{1}{r+p} - \bar{\varphi}_u - \frac{e^{k_2 x_u}}{k_2 \eta^+ - 1} C_2 - \frac{e^{k_3 x_u}}{k_3 \eta^+ - 1} C_3 \end{pmatrix}, \quad (4.8)$$

and k_0, k_1, k_2, k_3 are the roots of characteristic equation (3.23).

We also consider some limiting cases of formula (4.7). For single down barrier we let $x_u \rightarrow \infty$ and set $C_4 = C_5 \equiv 0$. As a result, first two equations in system (4.8) vanish. For single up barrier we let $x_d \rightarrow -\infty$ and set $C_6 = C_7 \equiv 0$. As a result, last two equations in system (4.8) vanish.

In presence of only positive jumps we let $\psi_3 \rightarrow -\infty$ and set $C_3 = C_7 = (1 - q) \equiv 0$. As a result, the first equation in system (4.8) vanishes. In presence of only negative jumps we let $\psi_0 \rightarrow \infty$ and set $C_0 = C_4 = q \equiv 0$. As a result, the fourth equation in system (4.8) vanishes.

5 Empirical Results

To compare our proposed pricing models, we calibrated them to the DAX implied volatility of 5 July, 2000. We used options with the middle-term maturity (6 month, $T \approx 0.46$). We obtained the following estimates for

- 1) double-exponential jumps-diffusion: $\hat{\sigma} = 0.18$, $\hat{\lambda} = 1.43$, $\hat{\eta}_u = 0.01$, $\hat{\eta}_d = 0.16$, $\hat{g} = 0.01$;
- 2) jump volatility model: $\hat{\sigma}_1 = 0.2$, $\hat{\sigma}_2 = 0.5$, $\hat{\lambda}_1 = 0.27$, $\hat{\lambda}_2 = 0.41$, $\hat{m}_1 = 0.25$, $\hat{m}_2 = -0.05$, $\hat{P}_1 = 0.77$.

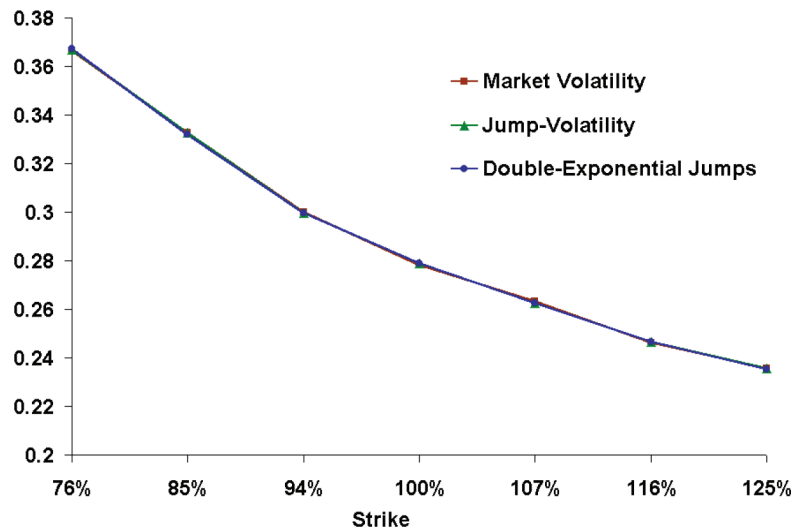


Figure 1: Market and model implied volatilities

Thus we see that both models perfectly fit the prices of vanillas and produce close prices for barrier options. We cannot accept or reject any of our models based on these observations. The final choice of the appropriate model should be based on empirical results of applying models for pricing and hedging barrier options in the real-life environment.

Conclusions

A straightforward *Extended Black-Scholes PIDE - Laplace Transform - OIDE - Numerical Inversion* method was applied to solve vanilla and double-barrier option pricing problems under double-exponential jump diffusion and jump volatility models. Both models allow obtaining simple close-form solutions consistent with the volatility smile. Numerical inversion of Laplace Transforms by Stehfest and Abate-Whitt algorithms is fast and precise procedure in the wide range of feasible financial parameters.

TABLE 1: PRICES OF VANILLA CALL AND DOUBLE-BARRIER KNOCK-OUT CALL

Strike	MIV	Call			DB-KO		
		BS	JV	DE	BS	JV	DE
3400	0.36671	1193.731	1193.77	1194.005	493.8065	907.0434	934.4318
3800	0.33272	853.6177	853.5831	853.0566	363.9345	622.9809	638.3344
4200	0.29993	551.4653	551.117	551.0405	228.9548	369.5617	378.3052
4500	0.27806	362.8873	363.819	364.0773	138.8061	216.3891	222.1931
4800	0.26310	220.9766	220.2191	220.4616	67.76316	105.4538	108.9901
5200	0.24633	97.41185	97.62025	97.67258	15.35449	24.47429	25.72025
5600	0.23558	36.7265	36.73566	36.69925	0.56699	0.97472	1.0618

We show market and model Black-Scholes implied volatilities in figure 1.

We see that both models can successfully be calibrated to the market data.

It is well recognized that pricing models can agree on prices of vanilla options, but they may widely disagree on prices of exotic options. To illustrate this issue, we first calculated prices of vanilla call (Call) using Black-Scholes formula with the market implied volatility (MIV) and our models, jump volatility (JV) and double-exponential jumps (DE), with parameters reported earlier. Then we calculated prices of double-barrier knock-out call (DB-KO) using formula (2.14) with the market implied volatility and our calibrated models. We used $S = 4483.03$, $T = 0.46$, $t = 0$, $S_d = 3200$, $S_u = 5800$, $r = 0.035$, $d = 0$ for different strikes. Our results are reported in table 1.

REFERENCES

- Abate J. and W. Whitt, (1995), "Numerical Inversion of Laplace Transforms of Probability Distributions," *ORSA Journal on Computing*, 7, 36–43.
- Black F. and M. Scholes, (1973), "The pricing of options and corporate liabilities," *Journal of Political Economy* 81, 637–659.
- Carslaw H.S. and J.C. Jaeger, (2000), "Conduction of Heat in Solids," *Reprint of the second edition, 1959, Clarendon Press.*
- Craddock M., D. Heath and E. Platen, (1999), "Numerical Inversion of Laplace Transforms: A Survey of Techniques with Applications to Derivative Pricing," *Research Paper 27, University of Technology, Sydney.*
- Hull J.C., (1995), "Options, Futures and Other Derivatives," *4-th Edition, Prentice Hall.*
- Geman H. and M.Yor, (1996), "Pricing and Hedging Double-Barrier Options: A Probabilistic Approach," *Mathematical Finance* 6, no 1, 365–378.

- Kou S., (2001), "Option Pricing under a Double Exponential Jump Diffusion Model," *Columbia University, Working Paper*.
- Lipton A., (2002), "Assets with Jumps," *Risk, September*, 149–153.
- Merton R., (1976), "Option pricing when underlying stock returns are discontinuous," *Journal of Financial Economics* 3, 125–144.
- Osband K., 2002–2003, *Finformatics*, *Wilmott Magazine*, No 1–5.
- Press W.H., S.A. Teukolsky, W.T. Vetterling and B.P. Flannary, (1992), "Numerical Recipes in C.," *Cambridge University Press*.
- Sepp A., (2003), "Pricing Path-Dependent Options under Jump-Diffusion Processes: Applications of Laplace Transform," *Working paper*, available at www.hot.ee/seppar.
- Skachkov I., (March 2002), "Black-Scholes Equation in Laplace Transform Domain," *Technical Articles on www.wilmott.com*.
- Stehfest H., (1970), "Algorithm 368 Numerical Inversion of Laplace Transforms," *Comm. ACM* 13, 479–490 (erratum 13, 624).
- Wilmott P., (1999), "Derivatives," *John Wiley & Sons*.

A Laplace transform

The Laplace transform is the integral

$$\bar{U}(x, p) := \mathcal{L}[U(x, \tau)] = \int_0^{\infty} e^{-p\tau} U(x, \tau) d\tau \quad (\text{A.1})$$

where p is a complex number whose real part is positive and large enough to make the integral convergent.

We use the following properties of Laplace transform

$$\mathcal{L}\left[\frac{\partial U}{\partial t}\right] = p\bar{U} - U(x, 0), \quad \mathcal{L}\left[\frac{\partial^n U}{\partial t^n}\right] = \frac{\partial^n \bar{U}}{\partial p^n}, \quad \mathcal{L}[H(t - t_0)U(t - t_0)] = e^{-pt_0}\bar{U}$$

where $H(t)$ is Heaviside's step function.

The following equation is known as Superposition principle, also as Duhamel's theorem.

$$\mathcal{L}\left[\int_0^t U_1(\tau)U_2(t - \tau)d\tau\right] = \bar{U}_1\bar{U}_2 \quad (\text{A.2})$$

If $\mathcal{L}[U(t)] = \bar{U}(p)$ then

$$\mathcal{L}\left[\frac{1}{\sqrt{\pi t}} \int_0^{\infty} e^{-\frac{x^2}{4t}} U(\tau)d\tau\right] = \frac{\bar{U}(\sqrt{p})}{\sqrt{p}}; \quad \mathcal{L}\left[\text{erfc}\left[\frac{x}{2\sqrt{vt}}\right]\right] = \frac{e^{-x\sqrt{p/v}}}{\sqrt{p}} \quad (\text{A.3})$$

where $\text{erfc}(x) = 2(1 - N(x))$, $N(x)$ is cumulative normal distribution function.

B VBA Code for Steffest Algorithm

To invert Laplace transform, we can use the algorithm by Stehfest (1970), which is one of the simplest and most powerful methods. If $U(x, p)$ is the Laplace transform of $V(x, \tau)$, then the original can approximately be

computed by

$$V(x, \tau) \approx \frac{\ln 2}{\tau} \sum_{j=1}^N Q_j U(x, j \frac{\ln 2}{\tau}) \quad (\text{B.1})$$

where coefficients Q_j are given by

$$Q_j = (-1)^{N/2+j} \sum_{k=(j+1)/2}^{\min\{j, N/2\}} \frac{k^{N/2} (2k)!}{(N/2 - k)! k! (k - 1)! (j - k)! (2k - 1)!} \quad (\text{B.2})$$

N is an even number and k is computed using integer arithmetic. The algorithm is very efficient and allows obtaining accuracy up to 8–10 significant digits. We found that the choice of $N = 10 - 20$ results in satisfactory accuracy.

To illustrate pricing in Laplace domain we provide VBA code for pricing vanilla call using algorithm by Stehfest.

```

`VanillaCall returns the BS price of vanilla call using numerical inversion of Laplace transform
Function VanillaCall(spot As Double, strike As Double, vol As Double, _
    tau As Double, rf As Double, Nterms As Integer) As Double
Dim x As Double, callprice As Double, p As Double, i As Integer, Coeffs As Variant
x = Log(spot / strike)
p = Log(2#) / tau
callprice = 0#
Coeffs = StehfestCoefficients(Nterms)
`Stehfest inversion formula
For i = 1 To Nterms
    callprice = callprice + Coeffs(i) * LaplaceCall(i * p, x, vol, rf)
Next i
callprice = p * callprice
VanillaCall = strike * callprice
End Function

`LaplaceCall returns the call price in Laplace solution domain for given transform parameter p
Function LaplaceCall(p As Double, x As Double, vol As Double, rf As Double) As Double
Dim C1 As Double, C2 As Double, psi1 As Double, psi2 As Double, zeta As Double
Dim drift As Double, vol2 As Double, laplacevalue As Double
vol2 = vol * vol
drift = rf - 0.5 * vol2
zeta = Sqr(drift * drift + 2# * vol2 * (rf + p))
psi1 = (-drift + zeta) / vol2 `Characteristic roots
psi2 = (-drift - zeta) / vol2
If x <= 0 Then
    C1 = (psi2 / (rf + p) + (1# - psi2) / p) / (psi1 - psi2)
    laplacevalue = C1 * Exp(psi1 * x)
Else
    C2 = (psi1 / (rf + p) + (1# - psi1) / p) / (psi1 - psi2)
    laplacevalue = C2 * Exp(psi2 * x) + Exp(x) / p - 1# / (rf
        + p)
End If
LaplaceCall = laplacevalue
End Function

`StehfestCoefficients returns coefficients in Stehfest inversion formula
Function StehfestCoefficients(Nterms As Integer) As Variant
Dim i As Integer, j As Integer, k As Integer, nh As Integer, sn As Integer, nm As Integer
nh = Nterms / 2 `Must be even
Nterms = 2 * nh
ReDim Coeffs(0 To Nterms) As Double, g(0 To Nterms) As Double, h(0 To nh) As Double
g(0) = 1
For i = 1 To Nterms
    g(i) = g(i - 1) * i
Next i
h(1) = 2# / g(nh - 1)
For i = 2 To nh
    h(i) = Exp(nh * Log(i)) * g(2 * i) / (g(nh - i) * g(i) * g(i - 1))
Next i
If nh Mod 2 = 0 Then sn = 1 Else sn = -1
For i = 1 To Nterms
    Coeffs(i) = 0
    If i < nh Then nm = i Else nm = nh
    j = Int((i + 1) / 2)
    For k = j To nm
        Coeffs(i) = Coeffs(i) + h(k) / (g(i - k) * g(2 * k - i))
    Next k
    sn = -sn
    Coeffs(i) = Coeffs(i) * sn
Next i
StehfestCoefficients = Coeffs
End Function

```